NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1420

ON THE CONTRIBUTION OF TURBULENT BOUNDARY LAYERS TO THE NOISE INSIDE A FUSELAGE

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G. M. Corcos
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Douglas Aircraft Company, Inc.

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*"Unedited by the NACA (the Committee takes no responsibility for the correctness of the author's statements.)"

**Aerodynamics Research Group
Douglas Aircraft Company, Inc.
Santa Monica, California

***California Institute of Technology
Pasadena, California
(Consultant, Aerodynamics Research Group, Douglas Aircraft Company, Inc.
Santa Monica, California)

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ABSTRACT

The following report deals in preliminary fashion with the transmission through a fuselage of random noise generated on the fuselage skin by a turbulent boundary layer. The concept of attenuation is abandoned and instead the problem is formulated as a sequence of two linear couplings: the turbulent boundary layer fluctuations excite the fuselage skin in lateral vibrations and the skin vibrations induce sound inside the fuselage. The techniques used are those required to determine the response of linear systems to random forcing functions of several variables. A certain degree of idealization has been resorted to. Thus the boundary layer is assumed locally homogeneous, the fuselage skin is assumed flat, unlined and free from axial loads and the "cabin" air is bounded only by the vibrating plate so that only outgoing waves are considered. Some of the details of the statistical description have been simplified in order to reveal the basic features of the problem.

The results, strictly applicable only to the limiting case of thin boundary layers, show that the sound pressure intensity is proportional to the square of the free stream density, the square of cabin air density and inversely proportional to the first power of the damping constant and to the second power of the plate density. The dependence on free stream velocity and boundary layer thickness cannot be given in general without a detailed knowledge of the characteristics of the pressure fluctuations in the boundary layer (in particular the frequency spectrum). For a flat spectrum the noise intensity depends on the fifth power of the velocity and the first power of the boundary layer thickness. This suggests that boundary layer removal is probably not an economical means of decreasing cabin noise.

In general, the analysis presented here only reduces the determination of cabin noise intensity to the measurement of the effect of any one of four variables (free stream velocity, boundary layer thickness, plate thickness or the characteristic velocity of propagation in the plate).

The plate generates noise by vibrating in resonance over a wide range of frequencies and increasing the damping constant is consequently an effective method of decreasing noise generation.

One of the main features of the results is that the relevent quantities upon which noise intensity depends are non-dimensional numbers in which boundary layer and plate properties enter as ratios. This is taken as an indication that in testing models of structures for boundary layer noise it is not sufficient to duplicate in the model the structural characteristics of the fuselage. One must match properly the characteristics of the exciting pressure fluctuations to that of the structure.

INTRODUCTION

In his efforts to minimize the noise levels for which he is responsible, the airplane designer has had to pay increasing attention to a source of noise which until recently had been ignored. This is the boundary layer. The boundary layer will generate noise whenever it is the seat of any fluctuating phenomenon. In particular it will nurture random pressure fluctuations whenever it is turbulent.

The designer's interest will naturally center on the characteristics of that part of the boundary layer noise which has been transmitted through the fuselage skin and into the cabin rather than on that part which is radiated into the free-stream. This is so because the radiation intensity is, as we shall see, a rapidly increasing function of the velocity of the boundary with respect to the air and to a likely observer outside a fuselage either the relative velocity of the plane is low (as near a take-off or landing) or the plane is considerably distant.

As a consequence the practical question which has to be raised concerns the effects on a fuselage skin, and on the air which it encloses, of the boundary layer pressure fluctuations acting directly on the skin.*

Two features of this problem are worth noting: To begin with, the fuselage will transmit noise only by deflecting laterally. The thickness of the skin is very small when compared to the wave length in metal of audible sound waves so that, effectively there are no fluctuating pressure gradients within the skin and hence the latter will not oscillate in lateral compression.

In the second place, the turbulent pressure fluctuations in the boundary layer are random both in space and in time. The fluctuations are generated locally. If they are measured simultaneously at two different points of the boundary layer, say on the skin itself, they are found to have no relationship with each other unless the two points are separated by a very short distance. Alike the value of the pressure fluctuation at a given point soon loses correlation with itself.**

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*Noise intensity is defined in this report as $\overline{p_1^2}/\rho_o a_1$. It is assumed that this is the physical quantity of interest. It has the dimensions of an energy flux; but it is not necessarily equal to the energy flux at some point in the field, nor is it necessarily equal to the density of energy radiated away (lost) by the fuselage.

**Recently two authors (Refs. 5 & 6) have suggested that the randomness in time is not independent of the randomness in space; i.e., that the pressure fluctuations at the wall are created by the convection at a single speed of a "frozen" pattern of pressure disturbances. Some attention is paid later to this eventuality which is treated as a special case.

We are thus led to visualize the process of transmission of boundary layer noise through the metal skin of the fuselage as follows: A multitude of external pressure pulses push the elastic skin in and out and the skin, in turn, not unlike a set of distributed pistons, creates inside the fuselage pressure waves which propagate and superimpose. This constitutes cabin noise.

It is of course desirable to determine the characteristics of this noise. One should point out that the data for the problem are not complete and are not likely to be so in the near future. Specifically the structure of the turbulent mechanism within the boundary layer and, in particular of the coupling between pressure fluctuations, velocity fluctuations and temperature fluctuations is not enough explained or measured to define wholly our forcing function. As a consequence it is not now possible to define say average cabin noise intensity $\overline{p_i^2}$ as a function of say, free stream Mach number, Reynolds number and plate characteristics. It is however possible and it is the purpose of this report to indicate the approximate functional dependence of $\overline{p_i^2}$ on these quantities and thus to give similarity rules which will reduce to a minimum the amount of testing required.

We assume at the outset that the boundary layer unsteady pressure field is known and that it induces small deflections in the skin. As a consequence

(a) The skin dynamics are described by a linear equation
(b) The generation of a random pressure field inside the cabin is a linear radiation problem.

Thus the mathematical techniques used are those required to obtain the response of linear systems to stochastic forcing functions of several variables.

We also assume the fuselage to be a large flat plate. This assumption is not necessary but it simplifies the discussion and allows us to present more clearly the new features of the problem.

The material in this report is presented as follows: First we study the radiation of sound from a randomly vibrating plate. It is found that the sound levels in the cabin are defined by the intensity and the scales of the plate normal accelerations.

Second, generalized Fourier analysis is put to use in order to relate the normal acceleration of the plate to the forces exerted on it by the boundary layer flow.

Third, the boundary layer forces are defined in terms of flow characteristics, and dimensional similarity is used to determine the significant parameters.

Finally, the functional form of the noise intensity in the cabin is given save for an unknown function of one non-dimensional parameter. This function depends on the frequency spectrum of the pressure fluctuations in the boundary layer. No measurements yielding this spectrum have been reported to date and speculations concerning it would introduce in the analysis both complication and uncertainty.

A summary of results is given at the end of the report. Some derivations and some of the longer arguments have been presented as appendices to the text.

I. THE ACOUSTIC COUPLING OF A RANDOMLY VIBRATING PLATE WITH AIR AT REST

We start with Rayleigh's well known solution of the acoustic equations when the sound is generated in an otherwise unbounded stationary gas by a large flat plate or disc oscillating normally to its plane (Ref. 1 page 107).

$$p_i = \frac{\rho_i}{2\pi} \int \frac{\partial V_n}{\partial t}\left(\vec{S},\ t - \frac{r}{a_i}\right) \frac{d\sigma}{|\vec{r}|} \tag{1}$$

where:

the static pressure p has been broken into a steady part p and a fluctuating part, p_i:

$p = \overline{p}(x,y,z) + p_i(x,y,z,t)$

V_n = the normal velocity of the plate

$d\sigma$ = an element of plate surface area

a_i = the speed of sound in the air

$\vec{S}$ = the vector position over which the integration is carried out

= (x',o,z')

$|\vec{r}|$ = the distance between source and field points

$= \sqrt{(x-x')^2 + y^2 + (z-z')^2} = r$

A = the total area of the plate

The subscript i refers to properties inside the cabin or on the side of the plate on which air is not flowing.

The normal acceleration $\partial Vn/\partial t$ is a random function of time and space and so is p_i. We wish to evaluate the mean square of the pressure (a quantity to which sound intensity is directly proportional).

We notice that for a sum

$$\overline{\left(\sum_1^n a_i\right)^2} = \overline{a_1^2} + \overline{a_2^2} + \overline{a_3^2} + \ldots + \overline{2a_1a_2} + \overline{2a_2a_3} + \overline{2a_3a_1}\ldots$$
$$= \overline{\sum_1^n a_i \sum_1^n a_j}$$

alike, for the integral in (1)*

*This discussion follows closely the arguments set forth in ref. 3 and the background of information on statistical methods can be found in ref. 2.

$$\overline{p_i^2} = \overline{\left(\frac{\rho_i}{2\pi}\int_A \frac{\partial V_n}{\partial t}\left(\vec{S}_1, t-\frac{r_1}{a_i}\right)\frac{d\sigma_1}{r_1}\right)\left(\frac{\rho_i}{2\pi}\int_A \frac{\partial V_n}{\partial t}\left(\vec{S}_2, t-\frac{r_2}{a_i}\right)\frac{d\sigma_2}{r_2}\right)}$$

and since $\overline{ax} = a\bar{x}$ if a does not depend on time, we can write

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2}\iint_{A\,A} \overline{\frac{\partial V_n}{\partial t}\left(\vec{S}_1, t-\frac{r_1}{a_i}\right)\frac{\partial V_n}{\partial t}\left(\vec{S}_2, t-\frac{r_2}{a_i}\right)}\frac{d\sigma_1}{r_1}\frac{d\sigma_2}{r_2} \tag{2}$$

Now the expression

$$\overline{\frac{\partial V_n}{\partial t}\left(\vec{S}_1, t-\frac{r_1}{a_i}\right)\frac{\partial V_n}{\partial t}\left(\vec{S}_2, t-\frac{r_2}{a_i}\right)}$$

is the correlation (i.e. the time average of the product) of the normal acceleration of the plate surface at two points $\vec{S_1}$ and $\vec{S_2}$ and at two different times $(t-r_1/a_i)$ and $(t-r_2/a_i)$. If the two points vibrate completely independently from each other, this expression is zero and if the two points are brought together ($S_1 \rightarrow S_2$, therefore $r_1 \rightarrow r_2$) the correlation function is simply $\overline{(\partial V_n/\partial t)^2}$. Now we assume that the average properties of the plate motion are the same anywhere on its surface and at any time (we assume statistical homogeneity and stationarity). Then the expression above for the correlation becomes merely a function of the distance ($|\vec{S_1} - \vec{S_2}|$) between the two points and of $\frac{(r_1-r_2)}{a_i}$ We call this function ψ :

$$\overline{\frac{\partial V_n}{\partial t}\left(\vec{S}_1, t-\frac{r_1}{a_i}\right)\frac{\partial V_n}{\partial t}\left(\vec{S}_2, t-\frac{r_2}{a_i}\right)} = \psi\left\{\left(|\vec{S}_1-\vec{S}_2|\right);\ \frac{r_1-r_2}{a_i}\right\}$$

then

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2}\iint_{A\,A}\frac{\psi\left\{|\vec{S}_1-\vec{S}_2|;\ \frac{r_1-r_2}{a_i}\right\}}{r_1 r_2}d\sigma_1\, d\sigma_2 \tag{3}$$

Now we can evaluate $\overline{p_i^2}$ (Y) under a variety of assumptions for ψ, for the plate area A and for the distance Y between the plate and the observer. We will consistently hold the view that the normal acceleration at most points of the plate surface are not correlated ($\psi = 0$) and that two points of the plate, in order to show appreciable correlation, must be a small fraction of the total plate size away from each other.

We define as λ the mean distance over which $(\partial V_n/\partial t)$ is strongly correlated (i.e. $\psi = 0\left[\overline{(\partial V_n/\partial t)^2}\right]$) and call it the integral (length) scale*.

* The integral scale is given, say in the x-direction by

$$\overline{(\partial V_n/\partial t)^2}\ \lambda_x = \int_{x_1}^{\infty}\overline{\frac{\partial V_n}{\partial t}(x_1,0,z_1)\frac{\partial V_n}{\partial t}(x_2,0,z_1)}\,dx_2$$

Our hypothesis can then be expressed as

$$\lambda \ll R$$

where R is the average linear dimension of the plate.

A representative case*

Suppose that the observer distance Y to the plate is such that

$$\lambda \ll Y \ll R$$

and that no appreciable phase difference can arise at Y between two strongly correlated signals (i.e., between sound pulses originating within λ of each other). Then, if we examined equation (3)

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2} \int_A \frac{1}{r_1} \left\{ \int_A \frac{\psi\left\{(|\vec{S}_1 - \vec{S}_2|), \frac{r_1 - r_2}{a_i}\right\} d\sigma_2}{r_2} \right\} d\sigma_1$$

we see that the inner integral contributes very little except when the point $\vec{S}_2$ is approximately within a distance λ of $\vec{S}_1$. Then, approximately $r_2 = r_1$ and $\psi \cong \overline{(\partial V_n/\partial t)^2}$. Thus the inner integral is approximately

$$\frac{\lambda^2 \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}}{r_1}$$

and equation (3) can be rewritten

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2} \lambda^2 \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \int_A \frac{d\sigma_1}{r_1^2} \tag{4}$$

We should note that

The integral in (4) is not finite if the plate area A is infinite

The length scale λ plays no role in the geometry of the problem.

The integral is a function of Y, the distance to the plate, and the plate dimensions only. For instance if the plate is circular and of radius R

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* This case is treated more rigorously in Appendix IA

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2}{2\pi} \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \log\left(1+\frac{R^2}{Y^2}\right) \tag{5}$$

4π

It is apparent from this result that the distance Y from the plate to the observer is measured in terms of plate diameters, and not in terms of average correlation lengths or wave lengths λ. This result holds for all the cases considered (see Appendix I) and depends only on the assumption that at a given time the plate vibrations are largely uncorrelated or incoherent.*

Here, in effect, λ^2 loses its identity and combines with $\overline{(\partial V_n/\partial t)^2}$ to define a strength.

Equation (4) indicates that in order to evaluate the pressure intensity for the case considered above, one must first determine $\overline{(\partial V_n/\partial t)^2}$ the mean square normal acceleration of the plate and λ, the length scale for the plate deflections. Other cases (i.e. cases for which either the plate deflects differently or the observer moves closer to it) are treated in Appendix IA. For some of them the time scale or mean period Θ is required as well as λ.

We rewrite Equation (4)

$$\left(\frac{\overline{p_i^2}}{\rho_i a_i}\right) = \frac{\rho_i}{a_i} \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \lambda^2 \, \Phi(g, Y) \tag{6}$$

where $\Phi(g, Y)$ is a function of the plate geometry and of the distance Y between the observer and the plate. It is defined from equation (4) as

$$\Phi(g, Y) = \frac{1}{4\pi^2} \int_A \frac{dx'\,dz'}{r^2}$$

Notice that if the integral scale is not the same in the x' and in the z' direction we may simply substitute in equation (6) $\lambda_{x'}$ and $\lambda_{z'}$ for λ.

* This result holds, as Appendix IB shows, even when the pressure is generated by the travel through air at rest of a "randomly bumpy plate" i.e., when the time-wise and space-wise variations of upwash are not independent. The latter example is therefore quite distinct from the flow of an infinite (periodically) wavy wall for which the only characteristic length is the wave length.

II. THE DYNAMIC BEHAVIOR OF THE SKIN

The response of the bare fuselage skin to the random pressure field of the boundary layer is given, in the absence of axial loads, by a plate equation. For a flat plate this equation is

$$\frac{Eh^2}{3\sigma(1-\mu^2)} \nabla^4 y + y_{tt} + \beta y_t = f(x,z,t) \tag{7}$$

where x and z are the coordinates along the plate surface (x being the free stream direction), y the deflection of the plate at a point, E the modulus of elasticity of the plate material, σ its density, μ Poisson's ratio, 2h the thickness of the plate, β a damping constant which has dimensions (1/T). Damping may be present because energy is absorbed either within the skin or by the air.* For air damping $\beta \sim \rho_i a_i / \sigma h$

Notice that to allow for air damping is to provide for a feedback in the coupling between the plate and the air at rest. On the other hand we exclude feedback between the plate and the boundary layer. In other terms we are not considering the possibility that the plate vibrations are large enough to induce time-dependent pressure gradients of the same order of magnitude as our forcing function. Such a feedback would amount to panel flutter. It cannot be handled by the present method.

$f(x,z,t)$ is the random force/unit mass exerted by the pressure fluctuations on the plate surface. It is characterized by a power spectral density $F(k_1,k_2,\omega)$ which is a continuous function of the wave numbers k_1 (in the x direction), k_2 (in the z direction) and of the frequency ω. The coefficient $E/3\sigma(1-\mu^2)$ has the dimensions of a velocity squared and it is defined as c^2.

---

*The skin construction may be such that the damping it causes is primarily viscous or primarily flexural. In the latter case it seems more appropriate to write with Ribner (ref. 5)

$$\frac{Eh^2}{3\sigma(1-\mu^2)} (1+i\beta_s)\nabla^4 y + y_{tt} + \beta_a y_t = f(x,z,t)$$

where β_s is the flexural damping constant due to the plate and β_a the damping due to the energy radiated to the air. As is shown in Appendix IB the noise intensity within the fuselage may or may not be related to the acoustic energy radiated by the plate and thus β_a may or may not be zero.

We still have to specify the space-wise boundary conditions on the plate and we are led, for the sake of simplicity, to either one of two limiting cases. In the first case, the forcing function (the random pressure field in the boundary layer) is characterized by an integral scale so large that at a given time, a skin area between two stiffeners (assumed rigid) is very likely to be subjected to a pressure load of the same sign (see Fig. Ia). This allows us to express y and f as functions of t only.

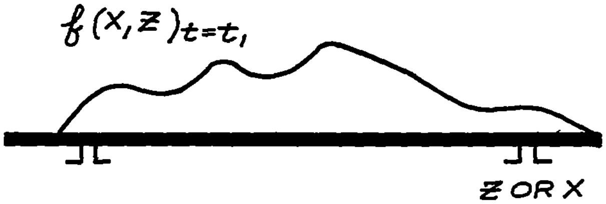


FIG 1a

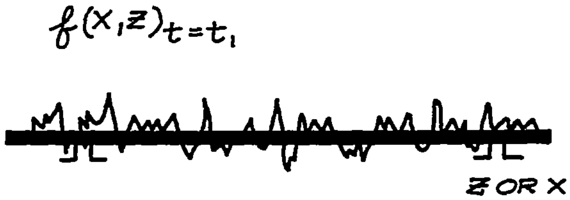


FIG. 1b

In the second case, the integral scale of the forcing function is very small in comparison with the distance between two stiffeners and the behavior of the skin is in the average very much as though the supports were removed to infinity (see Fig. Ib). The real case will in general be intermediate between these two limiting examples. However, the first case seems to apply to boundary layers of excessive thickness: A reasonable guess for the average correlation length might be one displacement thickness δ *; for δ* to be larger than the spacing between stiffeners (of the order of a foot) the boundary layer thickness δ would have to be of the order of five feet or more. This unlikely case is treated in reference (7).

On the other hand, the second limiting case (Fig. Ib) would seem to provide a reasonable model for boundary layer thickness not exceeding one foot. This is the model discussed now.

a) The Mean Acceleration

According to our assumption, the average motion

$$\frac{1}{A}\int_A \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\, d\sigma$$

is not sensibly affected by the presence of stiffeners. A large number of pulses act on the skin at a given time between two consecutive stiffeners. The random pulses may be positive or negative and thus there will be a large number of load reversals between supports. Then the effect of the boundary conditions can be expected to become small, in the average. Consequently one can define a generalized admittance and use it in much the same way as is often done in one-dimensional problems.* For instance, the mean square plate displacement is given by

$$\overline{y^2} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{dk_1 \, dk_2 \, d\omega \; \mathfrak{F}(k_1, k_2, \omega)}{\chi(k_1, k_2, \omega)}$$

Here the mean square of the forcing function f is related to the spectrum $\mathfrak{F}$ by:

$$\overline{\left[f(x,z,t)\right]^2} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \mathfrak{F}(k_1, k_2, \omega) \, dk_1 \, dk_2 \, d\omega$$

$1/\chi$ is the generalized admittance, and k_1, k_2 and ω are respectively the wave number in the x direction, the wave number in the z direction and the frequency. The determination of $1/\chi$ is easy once it is realized that this expression is the square of the Fourier transform of the fundamental solution and so can be written by inspection. Thus an average solution of (7) is:

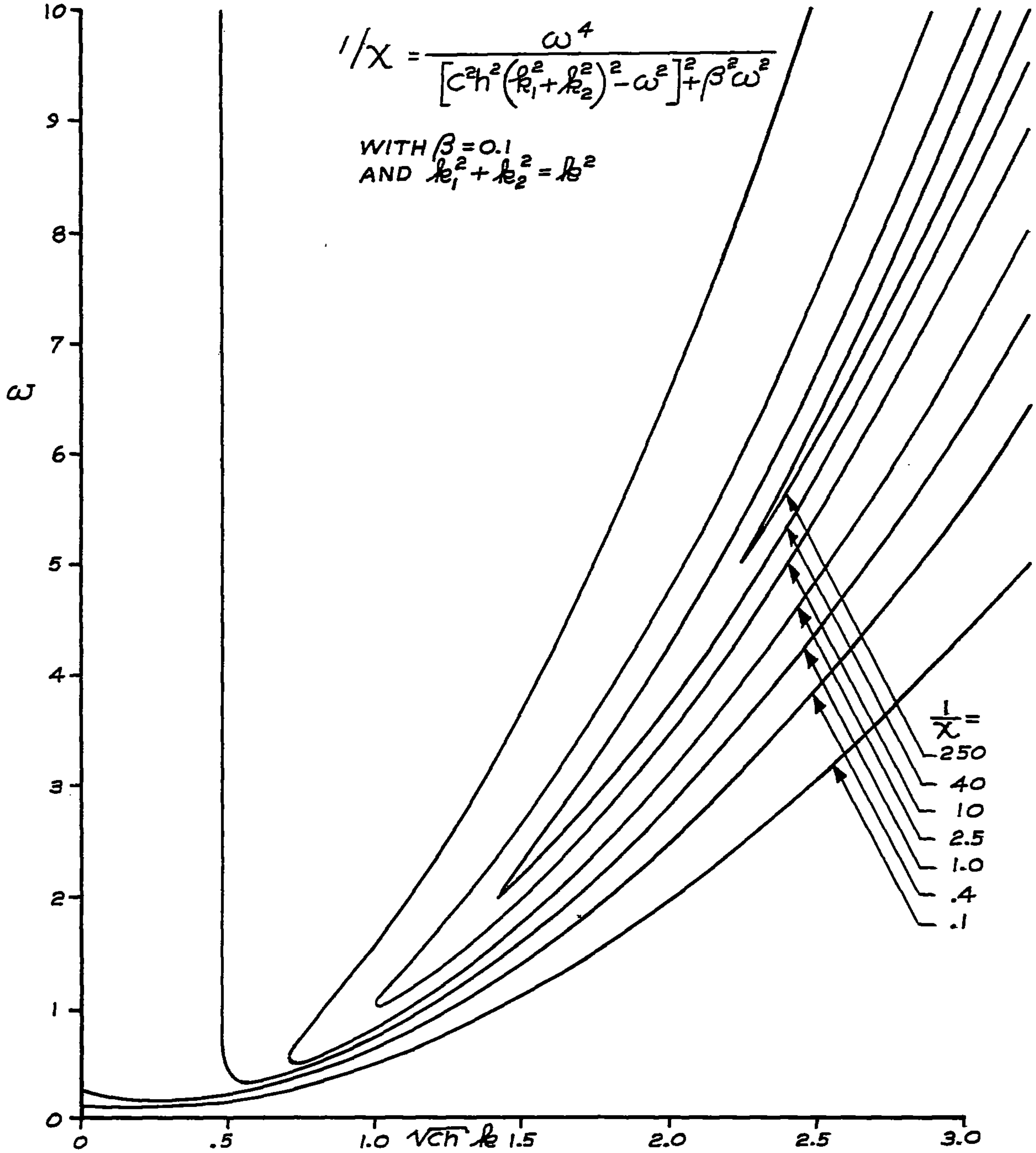
$$\overline{y^2} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\mathfrak{F}(k_1, k_2, \omega) \, dk_1 \, dk_2 \, d\omega}{\left[c^2 h^2 (k_1^2 + k_2^2)^2 - \omega^2\right]^2 + \beta^2 \omega^2} \tag{8}$$

and

$$\overline{\left(\frac{\partial V_m}{\partial t}\right)^2} = \overline{\left(\frac{\partial^2 y}{\partial t^2}\right)^2} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\mathfrak{F}(k_1, k_2, \omega) \, \omega^4 \, dk_1 \, dk_2 \, d\omega}{\left[c^2 h^2 (k_1^2 + k_2^2)^2 - \omega^2\right]^2 + \beta^2 \omega^2} \tag{9}$$

Equation (9) gives the mean square response of an unbounded plate to a random forcing function. One should notice that the plate will always exhibit resonance no matter what value the damping constant β may have. This resonance occurs, not at a given frequency or at a set of discrete

* See in particular reference (2).



THE ADMITTANCE MAP FOR AN UNBOUNDED PLATE

FIGURE 2

frequencies but over the whole frequency spectrum, whenever the following relationship obtains between frequencies and wave numbers:*

$$\omega^2 = c^2h^2\left(k_1^2 + k_2^2\right)^2$$

We can visualize the resonance condition as a crest or ridge in the wave space (see Fig. 2) which originates at the line $k^2 = \beta/ch\sqrt{2}$ and which becomes higher and steeper as the wave number and the frequency increase. Thus the effective damping is a function of the exciting frequency.

b) The Length Scale λ

Equation (6) shows that λ^2 is needed as well as $\overline{(\partial v_n/\partial t)^2}$. Now λ is a length scale. It was defined, say in the x direction as

$$\lambda_x \overline{\left(\frac{\partial v_n}{\partial t}\right)^2} = \int_{x_1}^{\infty} \overline{\left(\frac{\partial v_n}{\partial t}\right)_{\substack{x=x_1\\z=z_1}}\left(\frac{\partial v_n}{\partial t}\right)_{\substack{x=x_2\\z=z_1}}}\, dx_2$$

and could be termed the equivalent length of perfect correlation.

There are various ways of evaluating the integral scale. Perhaps the most convenient one for our purpose is that (found for instance in Ref. 2 Eq. II5) which is derived from the relationship between correlation functions and spectral functions. Thus if a stationary random function J(t) possesses a correlation function which is sufficiently well behaved,

$$\varphi(\tau) = \overline{J(t)J(t+\tau)}$$

---

* Here resonance is defined as the maximum of the response curve $1/\chi(k)$ holding ω constant. The locus of maximae holding k constant is given by:

$$\begin{cases} k^4 = \dfrac{\omega^2}{c^2h^2}\left[1 - \dfrac{\beta^2}{2c^2h^2k^4}\right] \\ chk^2 > \dfrac{\beta}{\sqrt{2}} \\ k^2 = k_1^2 + k_2^2 \end{cases}$$

These maximae correspond only for zero damping.

one can define a spectral density function

$$\frac{\pi}{2}\,\Phi(\omega) = \int_0^\infty \varphi(\tau)\cos\omega\tau\,d\tau$$

Now for the particular case $\omega = 0$, this gives

$$\frac{\pi}{2}\,\Phi(0) = \int_0^\infty \varphi(\tau)\,d\tau$$

Since

$$\lambda = \frac{\int_0^\infty \varphi(\tau)\,d\tau}{\varphi(0)}$$

we have the result that

$$\lambda = \frac{\pi}{2}\,\frac{\Phi(0)}{\varphi(0)} \tag{10}$$

We have already obtained a formal representation for the spectral density function of the plate. It is the integrand of Eq. (9) so we can write, in view of (10)

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\lambda_x = \frac{\pi}{2}\left[\int_0^\infty\int_{-\infty}^\infty \frac{\omega^4\,\mathcal{F}(k_1,k_2,\omega)\,dk_2\,d\omega}{\left[c^2h^2(k_1^2+k_2^2)^2-\omega^2\right]^2+\beta^2\omega^2}\right]_{k_1=0} \tag{11}$$

and a similar expression for λ_y.

$\Phi(0) = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}$

III. THE FORCING FUNCTION

We suppose that a turbulent boundary layer develops on the skin of the airplane (on one side of our plate). The forces which excite the plate are the pressure fluctuations experienced by the plate itself. We assume that all characteristics of the boundary layer are fixed once we have specified the boundary layer thickness δ, the free stream velocity U_∞ and density ρ_0. In terms of pressure fluctuations this implies that at a fixed point of the "wetted" surface of the skin, we have for the mean square pressure fluctuations

$$\overline{p_0^2} \sim \rho_0^2 U_\infty^4$$

and the integral scales, i.e.

$$l_x = \frac{\int_{x_1}^{\infty} \overline{p_0(x_1, 0, z_1) p_0(x_2, 0, z_1)}\, dx_2}{\overline{p_0^2}}$$

$$l_z = \frac{\int_{z_1}^{\infty} \overline{p_0(x_1, 0, z_1) p_0(x_1, 0, z_2)}\, dz_2}{\overline{p_0^2}}$$

are proportional to δ. Also the relative contribution to pressure intensity of the various frequency bands must be a function of U_∞, δ, and ρ_0 only so that if p_0 is a random function of three independent variables, x,z,t, $\overline{p_0^2}$ is related to a three-dimensional spectrum by:

$$\overline{p_0^2} = \int_0^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \pi(\omega, k_1, k_2)\, d\omega\, dk_1\, dk_2$$

such that:

$$\pi(\omega, k_1, k_2) = \rho_0 U_\infty^4 \frac{U_\infty}{\delta} \frac{1}{\delta} \frac{1}{\delta} F_2\left(\frac{\omega\delta}{U_\infty}, k_1\delta, k_2\delta\right)$$

Loosely speaking, this means that a characteristic frequency for pressure fluctuations is proportional to U_∞/δ and a characteristic wave length is proportional to δ. Now the forcing function of Eq. (7) is a force/unit mass so that according to our similarity hypothesis

$$\overline{f^2} \sim \frac{\rho_0^2 U_\infty^4}{\sigma^2 h^2}$$

One can thus define a spectral function associated with the forcing function f(x,z,t):

$$\overline{f^2} = \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \mathcal{F}_2(k_1, k_2, \omega)\, dk_1\, dk_2\, d\omega$$

and thus

$$\mathcal{F}_2(k_1, k_2, \omega) = \frac{\rho_0^2 U_\infty^4}{\sigma^2 h^2} \delta^2 \frac{\delta}{U_\infty} F_2(K_1, K_2, \Omega) \qquad (12)$$

Here F_2 is a function of K_1, K_2, and Ω only and these are non-dimensional variables:

$$\Omega = \omega\delta/U_\infty$$

$$K_1 = k_1\delta$$

$$K_2 = k_2\delta$$

In terms of these non-dimensional units, Eq. (8) becomes:

$$\overline{y^2} = \frac{\rho_0^2 U_\infty^4}{\sigma^2 h^2} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{dK_1\, dK_2\, d\Omega\, F_2(K_1, K_2, \Omega)}{\left[\frac{c^2 h^2}{\delta^4}\left(K_1^2 + K_2^2\right)^2 - \frac{U_\infty^2}{\delta^2}\Omega^2\right]^2 + \frac{\beta^2 U_\infty^2}{\delta^2}\Omega^2}$$

and Eq. (9) becomes

$$\overline{(y_{tt})^2} = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{\rho_o^2 U_\infty^8}{\sigma^2 h^2 \delta^4} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\Omega^4 F_2(K_1, K_2, \Omega)\, dK_1\, dK_2\, d\Omega}{\left[\frac{c^2h^2}{\delta^2}(K_1^2+K_2^2)^2 - \frac{U_\infty^2}{\delta^2}\Omega^2\right]^2 + \frac{\beta^2 U_\infty^2}{\delta^2}\Omega^2}$$

which can be written

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{\rho_o^2 U_\infty^4}{\sigma^2 h^2} \int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty \frac{\Omega^4 F_2(K_1, K_2, \Omega)\, dK_1\, dK_2\, d\Omega}{\left[\frac{c^2h^2}{\delta^2 U_\infty^2}(K_1^2+K_2^2)^2 - \Omega^2\right]^2 + \left[\frac{\beta\delta}{U_\infty}\Omega\right]^2} \tag{13}$$

Again, F_2 is a function of the integration variable only, so that one can write

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{\rho_o^2 U_\infty^4}{\sigma^2 h^2} H \left\{ \frac{ch}{\delta U_\infty} ; \quad \frac{\beta\delta}{U_\infty} \right\} \tag{14}$$

Equation (14) yields the two-non-dimensional parameters upon which the plate dynamics depend. The first one, $ch/\delta U_\infty$ is the product of a Mach number, $\left(\frac{\text{speed of free stream}}{\text{speed of propagation of waves in the plate}}\right)$ and a thickness ratio $\left(\frac{\text{boundary layer thickness}}{\text{plate thickness}}\right)$. The second one, $\beta\delta/U_\infty$ is a non-dimensional damping parameter which is, alike, a function of plate and free stream properties.

If we treat the equation for the integral scale (Eq. 11) in the same way, we notice that no new non-dimensional parameter occurs, so that, at most

$$\frac{\lambda}{\delta} = L \left\{ \frac{ch}{\delta U_\infty} ; \quad \frac{\beta\delta}{U_\infty} \right\} \tag{15}$$

We now wish to investigate the form of the functions H and L in Eqs. (14) and (15) respectively. First, we make an assumption which is not strictly necessary but which simplifies the manipulation of Eq. (12). We take the function $F(K_1, K_2, \Omega)$ to be symmetric in K_1 and K_2, which leads us to define a new wave number.

$$K = \sqrt{K_1^2 + K_2^2}$$

and to write

$$F_2(K_1, K_2, \Omega) = F(K, \Omega)$$

Thus, Eq. (13) becomes

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{2\pi \rho_0^2 U_\infty^4}{\sigma^2 h^2} \int_0^\infty \Omega^4 d\Omega \int_0^\infty \frac{K\, dK\, F(K, \Omega)}{\left[\frac{c^2h^2}{\delta^2 U_\infty^4} K^4 - \Omega^2\right]^2 + \left(\frac{\beta\delta}{U_\infty}\right)^2 \Omega^2}$$

Now, the damping parameter $\beta\delta/U_\infty$ is assumed small and under these circumstances it can be shown (see Appendix II) that

$$\int_0^\infty \frac{K\, dK}{\left[\frac{c^2h^2}{\delta^2 U_\infty^2} K^4 - \Omega^2\right]^2 + \frac{\beta^2\delta^2}{U_\infty^2}\Omega^2} \cong \int_0^\infty \frac{K\, dK}{4\Omega^2\left(\frac{ch}{\delta U_\infty} K^2 - \Omega\right)^2 + \left(\frac{\beta\delta}{U_\infty}\right)^2 \Omega^2} \tag{16}$$

The small difference between these two integrals can easily be evaluated for arbitrarily small values of $\beta\delta/U_\infty$ even though both integrals are unbounded as $\beta \to 0$. This leads us to believe that for low damping the main contribution to the inner integral comes from the resonance condition

$$K^2 = \frac{\delta U_\infty}{ch} \Omega$$

Thus, if the spectral function $F(\Omega, K)$ is reasonably wide, i.e. if $\partial F/\partial K \ll 1$ over a large range of K, Eq. (16) suggests that we write

$$F(K, \Omega) = F\left(\sqrt{\frac{\delta U_\infty}{ch}\Omega}, \Omega\right) \tag{17}$$

The requirement that F be flat in K when compared to 1/X(K) is equivalent to the requirement that the average correlation distance or integral scale for the boundary layer pressure fluctuations be small compared to integral scale of the plate deflection. Translated in physical terms the simplification suggested here is prompted by the following remark: If the plate has some stiffness, it makes little difference whether the forcing function is assumed to be distributed over small distances or made of concentrated loads (see Fig. 3)

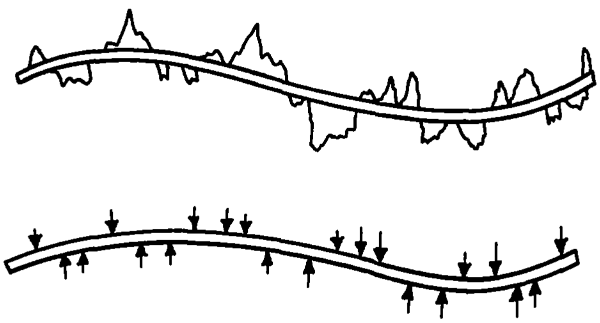


FIGURE 3

Thus a satisfactory model for the problem at hand would be the impact of rain drops on a metal roof. Equation (16) allows us to integrate over K, to get

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{2\pi \rho_0^2 U_\infty^4}{\sigma^2 h^2}\int_0^\infty \Omega^4 F\left(\sqrt{\frac{\delta U_\infty}{ch}}\,\Omega, \Omega\right)\left[\frac{\pi}{8\Omega^2}\,\frac{\delta U_\infty}{ch}\,\frac{U_\infty}{\beta\delta}\,d\Omega\right]$$

$$= \frac{\pi^2}{4}\,\frac{\rho_0^2 U_\infty^4}{\sigma^2 h^2}\left(\frac{\delta U_\infty}{ch}\right)\left(\frac{\beta\delta}{U_\infty}\right)\int_0^\infty \Omega^2 F\left(\sqrt{\frac{\delta U_\infty}{ch}}\,\Omega, \Omega\right)d\Omega$$

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{\rho_0^2 U_\infty^6}{\sigma^2 h^3 c\beta}\int_0^\infty \Omega^2\, F\left(\sqrt{\frac{\delta U_\infty}{ch}}\,\Omega, \Omega\right)\, d\Omega \tag{18}$$

The expression

$$\int_0^\infty \Omega^2 F\left(\sqrt{\frac{\delta U_\infty}{ch}}\Omega, \Omega\right) d\Omega$$

can be evaluated only when $F(K,\Omega)$ is known. It may be an increasing or a decreasing function of $\delta U_\infty/ch$. In the absence of data on the spectral function F, we will not attempt to define it.

The function H defined by Eq. (14) can be written:

$$H\left\{\frac{ch}{\delta U_\infty}; \frac{\beta\delta}{U_\infty}\right\} = \left(\frac{\beta\delta}{U_\infty}\right)^{-1} f_1\left(\frac{\delta U_\infty}{ch}\right) \tag{19}$$

where f_1, is an unspecified function related by (18) to the boundary layer pressure spectrum.

In order to determine $\overline{p_i^2}$ we need to find out, in addition, what quantities the integral scale λ depends on. Here we make use of considerations which are similar to those yielding Eq. (16) (see Appendix III). The result is that

$$\lambda \sim \left(\frac{ch\delta}{U_\infty}\right)^{\frac{1}{2}} f_2\left(\frac{\delta U_\infty}{ch}\right) \tag{20}$$

where f_2 is another function related to the boundary layer pressure spectrum by III(4). Now we are able to write Eq. (6) as

$$\overline{\frac{p_i^2}{\rho_i a_i}}(Y) \sim \frac{\rho_i}{a_i}\frac{\rho_0^2}{\sigma^2}\frac{U_\infty^5\delta}{h^2\beta}\Phi(Y,q)\, h\left(\frac{\delta U_\infty}{ch}\right) \tag{21}$$

Here

$$h = f_1 f_2^2$$

Expression (21) gives the functional dependence of pressure intensity "inside" on boundary layer parameters for a typical case. The only quantity, not immediately available is $h(\delta U_\infty/ch)$. It is probable that we shall have to await experimental data to define its numerical value reliably.

IV. SPECIAL CASES

1. Convected Turbulence

Two authors (ref. 5 & 6) have recently suggested that the boundary layer pressure fluctuations at any point of the fuselage skin are caused essentially by the passage over the point of a fixed (i.e. time independent) pattern of pressure disturbances carried downstream at a fixed convective velocity. So far, experimental evidence in proof or disproof is lacking. However, it is interesting to incorporate this special case in the general formulation which has been presented. Both the response of the plate and the coupling of the plate with the air at rest must then be reconsidered.

a) The coupling of the plate with air at rest in the case of convected turbulence

If a fixed spatial pressure distribution is carried downstream on the surface of the plate, it is easy to show that the (infinite) plate* response will be of the same kind, i.e., that it will consist of ripples which are randomly distributed in space but which travel through the plate at the same convective velocity as the boundary layer disturbance. The determination of the pressure field inside the fuselage is not in principle different for this case and has been carried out in Appendix IB**. The result is that for both subsonic moving ripples (with convection velocity $U_1 < a_i$) and moderately supersonic ones:

$$\overline{p_i^2} \sim \rho_i^2 \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \lambda^2 \int_A \frac{dx\,dz'}{r(r + M_i x')} \tag{I.10}$$

where

$$M_i = \frac{U_1}{a_i}$$

For higher supersonic speeds, the function of geometry and Mach number appearing as an integral is more complicated. The equation (I.10) above has the same form as equation (6). On the other hand there is a sharp difference in terms of energy radiated by the plate between the subsonic and the supersonic case, since no energy at all is radiated by subsonic ripples while the supersonic ones do generate some. One must, then, make

*Here the presence of transversal bulkheads will change the picture because of multiple reflections of the ripple.

**This problem can also be viewed as a steady (randomly bumpy) wing problem from the standpoint of a stationary observer.

a distinction between the results in terms of pressure intensity (the quantity of practical interest) and in terms of energy radiation. This distinction stems from the fact that (as is pointed out on page 7) the acoustical field investigated is truly a near field.

b) The response of the plate

According to the convective hypothesis, time is not an independent variable once the convective velocity U_1 is fixed. Translated in terms of the spectral density $\pi(\omega, k_1, k_2)$ of the pressure fluctuations, this means that $\pi(\omega, k_1, k_2)$ is zero, except when $\omega = U_1 k_1$, or in non-dimensional form, when $\Omega = (U_1/U_\infty)K_1$. We rewrite equation (12) for this special case.

$$\mathcal{F}(k_1, k_2, \omega) = \frac{\rho_0^2 U_\infty^4}{\sigma^2 h^2} \delta^2 \frac{\delta}{U_\infty} F(K_1, K_2, \Omega)\, \delta\left[\Omega - \frac{K_1 U_1}{U_\infty}\right]$$

Here $\delta\left[\Omega - K_1 \frac{U_1}{U_\infty}\right]$ is the Dirac delta function of the variable Ω. Then the plate response becomes

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} = \frac{4\rho_0^2 U_\infty^4}{\sigma^2 h^2} \int_0^\infty \int_0^\infty dK_1\, dK_2 \int_0^\infty \frac{\Omega^4 F(K_1, K_2, \Omega)\, \delta\left(\Omega - K_1 \frac{U_1}{U_\infty}\right) d\Omega}{\left[\frac{c^2 h^2}{U_\infty^2 \delta^2}(K_1^2 + K_2^2)^2 - \Omega^2\right]^2 + \frac{\beta^2 \delta^2}{U_\infty^2}\Omega^2} \tag{22}$$

$$= \frac{4\rho_0^2 U_\infty^4}{\sigma^2 h^2} \int_0^\infty \int_0^\infty \left(\frac{U_1}{U_\infty}\right)^4 \frac{K_1^4 F_1(K_1, K_2)\, dK_1\, dK_2}{\left[\frac{c^2 h^2}{\delta^2 U_\infty^2}(K_1^2 + K_2^2)^2 - \left(\frac{U_1}{U_\infty}\right)^2 K_1^2\right]^2 + \left(\frac{\beta^2 \delta^2}{U_\infty^2}\right)\left(\frac{K_1 U_1}{U_\infty}\right)^2} \tag{23}$$

where

$$F_1(K_1, K_2) = F\left(K_1, K_2, K_1 \frac{U_1}{U_\infty}\right)$$

Now if we assume as before that F is symmetric in K_1 and K_2 and substitute

$$\begin{cases} K = \sqrt{K_1^2 + K_2^2} \\ \theta = \tan^{-1} \dfrac{K_2}{K_1} \end{cases}$$

we finally get

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \cong \frac{\rho_o^2 U_\infty^8 \delta^2 B^2}{\sigma^2 h^2 c^3 \beta} \int_0^{2\pi} \cos^4\theta \; F_2\left(B, \cos\theta, \frac{\delta U_\infty}{ch}\right) d\theta \tag{24}$$

Here

$$B = U_1 / U_\infty$$

and

$$F_2\left(B, \cos\theta, \frac{\delta U_\infty}{ch}\right) = F\left(K_1, K_2, K_1 \frac{U_1}{U_\infty}\right)$$

with

$$K_1^2 + K_2^2 = K^2$$

and

$$K = \frac{\delta U_\infty}{ch} B\cos\theta$$

The length scale λ is indicated by the following dimensional argument.

The mean correlation length or integral scale is a weighted average of all wave lengths, so that dimensionally

$$\lambda \sim \frac{1}{k}$$

Since resonance dominates the plate response, $\frac{1}{k}$ is given from the plate response equation (equation 23) by the resonance condition

$$(k\delta)^2 = K^2 = \left(\frac{U_1}{U_\infty}\right)^2 \cos^2\theta \left(\frac{\delta U_\infty}{ch}\right)^2$$

or

$$\lambda \sim \frac{1}{k} \sim \frac{ch}{U_\infty} \tag{25}$$

A similar reasoning would have yielded, in the non-convective case, $\lambda \sim \left(ch\delta/U_\infty\right)^{1/2}$ instead of eq. (20).

Combining (24) and (25) according to (I.10) we notice that we can still write as in equation (21).

$$\frac{\overline{p_i^2}}{\rho_i a_i} \sim \frac{\rho_i}{a_i} \frac{\rho_o^2}{\sigma^2} \frac{U_\infty^5 \delta}{h^2 \beta} \Phi_1(\gamma, g, M) \, h_1\left(\frac{\delta U_\infty}{ch}\right) \tag{26}$$

Here Φ_1 is a weak function of the Mach number as seen from (I.10).

2. <u>The case of zero scale</u>

Under some circumstances it is possible that the space average of the plate motion vanishes, i.e.:

$$\int_0^\infty \overline{\frac{\partial V_n}{\partial t}(\vec{S_1},0)\,\frac{\partial V_n}{\partial t}(\vec{S_2},0)}\,dA = 0$$

This does not mean that the normal accelerations at two neighboring points show no correlation, but that the correlation function becomes negative as indicated in Fig. (4) and in such a way that its space integral vanishes. We can then consider the normal accelerations as dipoles rather than sources and we are led to a slightly different radiation problem. Appendix IC shows, however, that if one defines a length λ'_ρ such that

$$\lambda'^2_\rho = \int_0^\infty \rho R_\rho \, d\rho$$

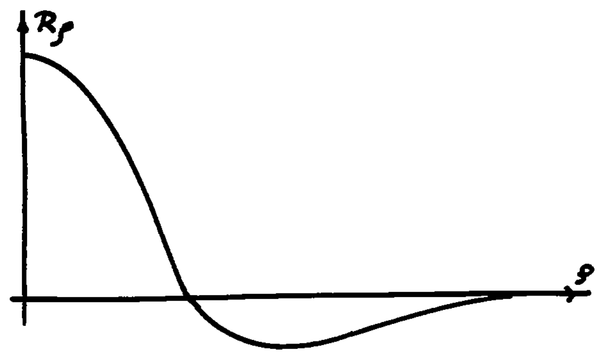


FIGURE 4

The results are again identical in form with those of equation (6). Here λ' can be viewed as the mean moment arm of deflection moments. Alternatively one can redefine the integral scale as

$$\lambda'_\rho = c \int_0^\infty |R_\rho| \, d\rho \tag{27}$$

where c is a constant. Equation (27) can thus be used to define the integral scale in any event.

V. SUMMARY OF RESULTS AND DISCUSSION

Appendix I discusses in addition to the cases mentioned in the text a few examples which provide different limiting conditions. Thus the observer is brought close to the plate ($Y \ll \lambda$). A short time scale is considered etc..... The common feature of all these analyses is that the resulting mean noise intensity can always be represented, say by equation (26). We shall therefore retain this equation:

$$\frac{\overline{p_i^2}(Y)}{\rho_i a_i} \sim \frac{\rho_i}{a_i} \frac{\rho_0^2}{\sigma^2} \frac{U_\infty^5 \delta}{h^2 \beta} \Phi(Y, g, M_i)\, S\left(\frac{\delta U_\infty}{ch}\right)$$

as the most general statement that we can make at the present time. Here

$\overline{p_i^2}$ = mean square noise intensity inside

ρ_i = air density inside

a_i = speed of sound inside

ρ_0 = air density in the free stream

σ = plate density

U_∞ = free stream velocity

δ = boundary layer thickness

$2h$ = plate thickness

β = viscous damping constant (of units 1/time)

Y = perpendicular distance between observer and fuselage

g = geometry of the plate

M_i = Mach number U_1/a_i

U_1 = convective velocity of turbulence pattern

c = characteristic velocity in the plate = $\sqrt{\dfrac{E}{3\sigma(1-\mu^2)}}$

For all but high supersonic velocities, the dependence of Φ on M_1 is quite small and can be disregarded. The function Φ (Y,g), a quantity which does not depend on the dynamics of the problem but only on its geometry should be modified to take into account the fact that the fuselage is a cylinder and not a large flat plate.

The form of the function S cannot be given here both because no information is yet available on boundary layer pressure spectra and because S depends too critically on the type of model assumed. However, if $\overline{p_i^2}$ is measured while any one of the four variables defining S (δ, U_∞, c or h) is varied, then the functional form of the noise intensity inside a fuselage can be determined. Thus the main contribution of the analysis is to diminish the extent of the testing required.

One of the conclusions which can be drawn from the foregoing equation is that unless the boundary layer pressure spectrum is a very sharp function of frequency (which would make S very sensitive to $\delta U_\infty/ch$) it is not practical to decrease cabin noise by boundary layer suction: Since the noise intensity is a weak function of boundary layer thickness, decreasing appreciably cabin noise would involve the removal of a prohibitive amount of air.

Another conclusion is that increasing the damping is a very effective way of limiting the production of noise of all frequencies, since the structure transmits sounds essentially by resonance.

The analysis which has been presented deliberately omitted some of the features of the problem which would influence the results and introduce new parameters. For instance, the fuselage of commercial airplanes is usually subjected to an axial tension as well as other loads. In addition the skin is curved. To account for these features of the problem one would introduce further terms in the differential equation describing the plate and one could treat it in much the same way as has been done here.

The general methods which have been used are adaptable in addition to the study of a germane problem, the fatigue of panels which are buffeted by a turbulent boundary layer.

A NOTE ON TESTING

The discussion of the various limiting solutions makes it clear that for the transmission of boundary layer noise through a structure, the ratio of outside (boundary layer) noise to inside (cabin) noise is in general a function of boundary layer as well as structural characteristics.

This is to say, first, that an attenuation coefficient cannot be defined by testing the structure alone with a standard noise source. Thus accurate testing requires at the outset that the model be tested for transmission of a noise similar to boundary layer noise. The main property of such a noise, as we have seen is that it must be random in space as well as in time, which precludes the use of one or a few concentrated sources as noise generators. The only proper substitutes for boundary layer pressure fluctuations are forcing functions whose effects on a fuselage are local.* The impact of water drops for instance might be found adequate simulation. Further, similarity in testing requires the matching of parameters which are ratios of plate and forcing function properties. For instance if the forcing function used in the test is a turbulent boundary layer, similarity parameters are:

$$\frac{a}{\delta} \ \& \ \frac{b}{\delta} \ ; \quad \frac{\beta\delta}{U_\infty} \ ; \ \frac{U_\infty}{a_i} \ ; \ \frac{\delta}{Y} \ ; \ \frac{ch}{\delta U_\infty} \ ;$$

*This is not true of jet noise which is generated away from the fuselage.

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APPENDIX I

THE RANDOM RADIATION OF A PLANE SURFACE:

A. FOUR LIMITING CASES

In order to determine the coupling between fuselage vibrations and cabin air one has to choose a model for the correlation ψ between the normal accelerations at two different points of the plate. The model which was discussed and for which equation (4) was made plausible is predicated upon two conditions:

A. That the observer is distant enough so that a large number of plate elements vibrating independently contribute sound in comparable amounts, i.e.

$$\lambda \ll Y$$

Here as before, λ is the integral (length) scale for the plate normal accelerations and Y is the perpendicular distance between the observer and the plate.

B. That the time scale of the phenomenon is large enough so that the differences in phase (introduced by the unequal distance from the point at Y to the various points of a plate element of length λ) are unimportant, i.e.

$$\lambda \ll a_1 \Theta$$

a_1 is the speed of sound in the fuselage air, and Θ is the integral (time) scale for the phenomenon:

$$\Theta = \frac{\int_{t_1}^{\infty} \overline{\frac{\partial V_n}{\partial t}(x_1, z_1, t_1) \frac{\partial V_n}{\partial t}(x_1, z_1, t_2)}\, dt_2}{\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}}$$

Then one can choose a simple model for the correlation function ψ

$$\psi(\xi, \eta, \tau) = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \lambda^2 \delta(\xi) \delta(\eta) \varphi(\tau) \qquad \text{(I.1)}$$

where δ is the delta function. The normal accelerations are assumed perfectly correlated within a length λ and not at all for distances greater than λ . Then

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2 \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}}{4\pi^2} \iint_{A\,A} \frac{dx_1'\, dx_2'\, dz_1'\, dz_2'}{r_1\, r_2}\, \delta(x_1'-x_2')\,\delta(z_1'-z_2')\,\varphi\left(\frac{r_1-r_2}{a_i}\right)$$

and upon integrating

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2}{4\pi^2} \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \iint \frac{dx'\,dz'}{r^2}$$

which is equation (4). This case, (1) $\lambda \ll Y$; 2) $\lambda \ll a_i\Theta$) corresponds to the following conditions. The passenger (or the microphone) is far from the plate (in terms of λ), the boundary layer is thick and the airplane velocities low. One may well wonder about cases for which these conditions do not apply. While it appears difficult to answer such a query with generality it is possible to consider other limiting cases.

For instance let us assume that condition 2 still applies but that our observer is extremely close to the plate. This would correspond to the following physical case: A thin fuselage skin, a thick boundary layer, a low airplane velocity and we are measuring noise by placing a microphone very close to the skin and insulating it on all sides except the side which faces the skin. Then $\lambda \gg Y$; $\lambda \ll a_i\Theta$.

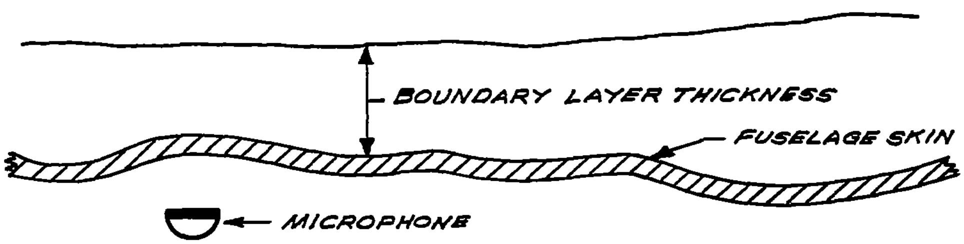

FIGURE 5

Under these conditions the noise at the microphone is contributed primarily from a single plate element which in the average vibrates in phase. The evaluation of this contribution is particularly simple. We can write, very nearly

$$\overline{\frac{\partial V_n}{\partial t}\left(x_1', z_1', t-\frac{r_1}{a_i}\right)\frac{\partial V_n}{\partial t}\left(x_2', z_2', t-\frac{r_2}{a_i}\right)} = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \tag{I.2}$$

and

$$\overline{p_i^2} = \frac{\rho_i}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\int_0^\lambda\int_0^\lambda \frac{\rho_1\,d\rho_1\,\rho_2\,d\rho_2\,d\theta_1\,d\theta_2}{r_1 r_2}$$

If, for the sake of definiteness, we assume the element circular, then

$$\overline{p_i^2} = \rho_i^2\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\left(\sqrt{\lambda^2+Y^2}-Y\right)^2$$

and since $\lambda \gg Y$ it is permissible to write

$$\overline{p_i^2} \cong \lambda^2\rho_i^2\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}$$

The pressure intensity is therefore given as

$$\frac{\overline{p_i^2}}{\rho_i a_i} \cong \frac{\rho_i}{a_i}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\lambda^2 \tag{I.2}$$

Thus Eq. (6) applies for the very close as well as for the very far field when phase effects are not important ($\lambda \ll \Theta a_1$).

Now assume that we carry on the same experiment but that the boundary layer is thin and that the velocity of the airplane is high so that the exciting frequencies are high. Let us assume in addition that the skin is thick, so that $Y \ll \lambda;\ |\Theta| \ll \lambda/a_i$.

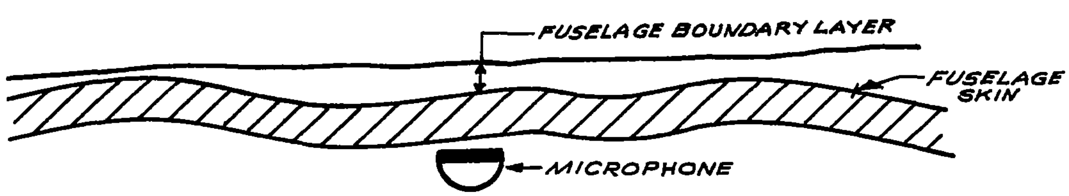

FIGURE 6

Now the time scale of the plate motion is short and phase effects are prevalent. We define a simple time history in analogy to the space description of Eq. (I.1)

$$\overline{\frac{\partial V_n}{\partial t}(x_1', z_1', t - \frac{r_1}{a_i}) \frac{\partial V_n}{\partial t}(x_2', z_2', t - \frac{r_2}{a_i})} = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \delta\left(\frac{r_1 - r_2}{a_i}\right) \Theta$$

The microphone still receives signals effectively only from one plate element and all points within that element vibrate in phase but the pressure pulses originating from that element do not arrive at the microphone in the same time. Then:

$$\overline{p_i^2} = \frac{\rho_i^2 \Theta}{4\pi^2} \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \int_{A'} \int_{A'} \delta\left(\frac{r_1 - r_2}{a_i}\right) \frac{ds_1 \, ds_2}{r_1 r_2} \tag{I.3}$$

A' is simply λ. Equation (I.3) is evaluated by noticing that:

$$\int_a^b f(\xi) \, \delta[g(\xi)] \, d\xi = \frac{\sum_i f(\xi_i)}{|g'(\xi_i)|} \tag{I.4}$$

Here ξ_i are the real roots of $g(\xi) = 0$ which are included in the interval between and and b. We only have one root, namely $r_1 = r_2$. If we choose to integrate, say, with respect to s_2 first we get (assuming again that the element is circular

$$\overline{p_i^2} = \frac{\rho_i^2}{2\pi} \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \Theta \int_{A'} \frac{ds_1}{r_1} \int_{A'} \delta\left(\frac{r_1 - r_2}{a_i}\right) \frac{\rho_2 \, d\rho_2}{\rho_2^2 + Y_2^2} \tag{I.5}$$

and according to (I.4) the inner integral yields:

$$\int_0^\lambda \delta\left(\frac{r_1 - r_2}{a_i}\right) \frac{\rho_2 \, d\rho_2}{r_2}$$

$$= \left[\frac{\rho_2}{r_2}\right]_{\rho_2 = \rho_1} \left[\frac{r_2 a_i}{\rho_2}\right] = a_i$$

so that

$$\overline{p_i^2} = \rho_i^2 a_i \Theta \, \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \int_0^\lambda \frac{\rho \, d\rho}{r}$$

$$= \rho_i^2 a_i \Theta \, \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \left\{ \sqrt{\lambda^2 + Y^2} - Y \right\}$$

$$\overline{p_i^2} \cong \rho_i^2 a_i \Theta \lambda \, \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \tag{I.6}$$

The time scale appears explicitly in the answer. For the unbounded plate however it is simply proportional to δ / U_∞ just as the time scale for the boundary layer pressure fluctuations.

Finally we may consider a physical case for which phase effects are important and for which the microphone has been placed a large distance away from the plate. i.e.: $\Theta \ll \lambda / a_i$; $\lambda \ll Y$

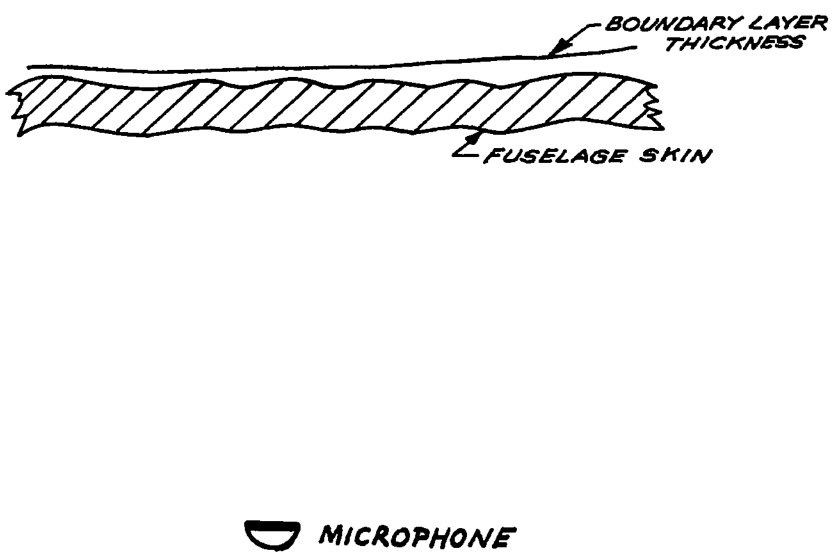

FIGURE 7

Now the contribution from each sub-element of vibrating plate is still in the average independent from that of the next one. However, there are in addition cancellations from within one element just as in the previous case. This will happen if the boundary layer is thin, the airplane velocity is high (Θ small) and the observer is far from the fuselage wall.

In order to evaluate this limiting case we first specify the time behavior of the correlation function: we write

$$\overline{\frac{\partial V_n}{\partial t}\left(\rho_1, \theta_1, t_1 - \frac{r_1}{a_i}\right) \frac{\partial V_n}{\partial t}\left(\rho_2, \theta_2, t_1 - \frac{r_2}{a_i}\right)} = \Psi\left(\rho_1 - \rho_2; \theta_1 - \theta_2; \frac{r_1 - r_2}{a_i}\right)$$

$$= \psi(\rho_1 - \rho_2; \theta_1 - \theta_2)\,\delta\left(\frac{r_1 - r_2}{a_i}\right)\Theta(\rho_1 - \rho_2, \theta_1 - \theta_2) \qquad \text{(I.7)}$$

so that

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2}\int_A\int_A \psi(\rho_1 - \rho_2; \theta_1 - \theta_2)\,\delta\left(\frac{r_1 - r_2}{a_i}\right)\Theta(\rho_1 - \rho_2, \theta_1 - \theta_2)\, ds_1\, ds_2$$

and we integrate first with respect to ρ_2. Using the same techniques as in the previous example, we get:

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2}\int_A\int_0^{2\pi} \frac{\psi(0, \theta_1 - \theta_2)\,\Theta(0, \theta_1 - \theta_2)\, a_i}{r_1}\, ds_1\, d\theta_2$$

Now we assume that

$$\psi(0, \theta_1 - \theta_2) = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\,\delta\left\{\rho_1(\theta_1 - \theta_2)\right\}\lambda(0)$$

Integrating with respect to θ_2, θ_1, ρ_1 successively:

$$\overline{p_i^2} = \frac{\rho_i^2}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\Theta(0)\,\lambda(0)\, a_i \int_A \frac{d\rho_1}{r_1}\, d\theta_1$$

For a circular plate of radius R, this would give

$$\overline{p_i^2} = \frac{\rho_i^2}{2\pi}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\Theta(0)\,\lambda(0)\,a_i\int_0^R \frac{d\rho}{r}\,d\theta$$

$$\cong \frac{\rho_i^2}{2\pi}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\Theta\,a_i\,\lambda\,\ln\left[1+\frac{R}{Y}\right]$$

In general, and defining

$$h(q, Y_2) = \int_A \frac{d\rho\,d\theta}{r}$$

a function of the plate geometry and of the distance Y only, we have

$$\frac{\overline{p_i^2}}{\rho_i a_i} \sim \rho_i\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\Theta\,\lambda\,h(q, Y) \tag{I.8}$$

APPENDIX I

B. THE NOISE GENERATED BY SKIN RIPPLES OF FIXED VELOCITY:

If the turbulence pattern is frozen, as discussed in section IV-1 ripples will travel through the (infinite) skin at a fixed convective velocity. Then the correlation function ψ must be written differently:

$$\psi(\xi,\eta,\tau) = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2} \lambda^2 \delta(\eta)\,\delta(\xi + U_1\tau)$$

where U_1 is the speed of propagation of the ripple (turbulence convective speed) and therefore

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\iint \frac{dx_1' dx_2' dz_1' dz_2'\,\delta(z_1' - z_2')\,\delta(x_1' + M_1 r_1 - x_2' - M_1 r_2)}{r_1 r_2}$$

or

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\int \frac{dx_1' dz_1'}{r_1}\int \frac{dx_2'}{r_2}\,\delta\left[(x_2' + M_1 r_2) - (x_1' + M_1 r_1)\right]$$

where now

$$r_1 = \sqrt{x_1'^2 + z_1'^2 + Y^2}$$

$$r_2 = \sqrt{x_2'^2 + z_1'^2 + Y^2}$$

The inner integral is of the form

$$\int f(\xi)\,\delta\left[g(\xi)\right] d\xi$$

which can be written

$$\sum \frac{f_i(\xi)}{g_i'(\xi)}$$

as in part A. The expression

$$g_i(\xi) = \left[(x_2' + M_1 r_2) - (x_1' + M_1 r_1)\right] \qquad \text{(I.9)}$$

has either one or two real roots depending as $M_1 < 1$ or $M_1 > 1$ respectively. For $M_1 < 1$, the only real root is

$$x_1' = x_2'$$
$$r_1 = r_2$$
$$g'(\xi) = 1 + \frac{M_1 x_1'}{r_1}$$

and thus the inner integral yields

$$\frac{1}{r_1 + M_1 x_1'}$$

so that:

$$\overline{(p_i^2)} = \frac{\lambda^2 \rho_i^2}{4\pi^2} \left(\overline{\frac{\partial V_n}{\partial t}} \right)^2 \int \frac{dx_1' dz_1'}{r_1 (r_1 + M_1 x_1')} \qquad \text{(I.10a)}$$

notice that equation (I.10a) above tends to equation (4) for low convective speeds.

For $M_1 > 1$, (I.9) has, in addition to the root $x_1' = x_2'$, another root given by

$$r_2 = \frac{(M_1^2 + 1) r_1 + 2 M_1 x_1'}{M_1^2 - 1}$$

It is easy to show that this root exists for all values of x_1'. In order to simplify the integration let us assume slightly supersonic conditions; i.e. let us write

$$M_1 = 1 + \varepsilon$$

where

$$\varepsilon << 1$$

Then

$$r_2 \cong \frac{(1+\varepsilon)\left[r_1 + x_1'\right]}{\varepsilon}$$

and

$$x_2' \cong -\frac{1+\varepsilon}{\varepsilon}\left(r_1 + x_1'\right)$$

and the inner integral =

$$\frac{1}{r_2 + M_1 x_2'}\Bigg]_{\substack{r_2 = r_i \\ x_2' = x_i'}} = \frac{1}{M_1(r_1 + M_1 x_1')}$$

so that for the supersonic case:

$$\overline{p_i^2} = \frac{\lambda^2 \rho_i^2 \overline{\left(\frac{\partial v_n}{\partial t}\right)^2}}{4\pi^2} \frac{M_1 + 1}{M_1} \int \frac{dx_1' dz_1'}{r_1 (r_1 + M_1 x_1')} \tag{I.10b}$$

A result which is save for a constant coefficient the same as I.10a.

APPENDIX I

C. THE GENERATION OF NOISE BY PLATE DEFLECTIONS OF ZERO SCALE

If the space average of the correlation function is zero and if the plate vibrations are isotropic in x' and z' one can define a new length scale as a moment arm:

$$\lambda'^2 = \int_0^\infty \rho\,\psi(\rho, 0)\,d\rho$$

$$\rho = (|\vec{S}_1 - \vec{S}_2|) \qquad (\vec{S}_1 \text{ is a fixed point})$$

alternatively one can define a modified integral scale

$$\lambda'' = \int_0^\infty |\psi(\rho, 0)|\,d\rho$$

Here

$$\lambda'' = c\lambda'$$

where c is a constant.

Then, one can idealize the correlation function as

$$\psi(|\vec{S}_1 - \vec{S}_2|t) = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\,\lambda'^2\,\delta(\theta_1 - \theta_2)\,\delta'(\rho_1 - \rho_2)\,\varphi(t)$$

provided

$$\lambda' \ll r \quad ; \quad \lambda' \ll a_i$$

It follows that

$$\overline{p_i^2} = \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\frac{\lambda'^2 \rho_i^2}{4\pi^2} \iint_{AA} \frac{\varphi\left(\frac{r_1 - r_2}{a_i}\right)\delta'(\rho_1 - \rho_2)\,\delta(\theta_1 - \theta_2)\,dx_1'\,dz_1'\,dx_2'\,dz_2'}{r_1 r_2}$$

and integrating with respect to θ_1

$$= \overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\,\frac{\lambda'^2 \rho_i^2}{4\pi^2} \int_A \frac{\rho_2\,d\rho_2\,d\theta_2}{r_2} \int_0^\infty \frac{\rho_1\,d\rho_1}{r_1}\,\varphi\left(\frac{r_1 - r_2}{a_i}\right)\delta'(\rho_1 - \rho_2)$$

Now, unless a or b = 0

$$\int_a^b f(\xi)\,\delta'(\xi)\,d\xi = -\int_a^b f'(\xi)\,\delta(\xi)\,d\xi$$

and thus

$$\overline{p_i^2} = -\frac{\lambda'^2 \rho_i^2}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\int_A \frac{\rho_2 \, d\rho_2 \, d\theta_2}{r_2}\int_0^\infty \delta(\rho_1-\rho_2)\frac{\partial}{\partial\rho_1}\left[\frac{\rho_1}{r_1}\varphi\left(\frac{r_1-r_2}{a_i}\right)\right]d\rho_1$$

$$= \frac{\lambda'^2 \rho_i^2}{4\pi^2}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\int_A \varphi(0)\left[\frac{\rho\, d\rho}{r^2} - \frac{\rho^3 d\rho}{r^4}\right] + \varphi'(0)\left[\frac{\rho^3 d\rho}{r^3 a_i}\right]$$

If there is a (time) microscale

$$\varphi'(0) = 0$$

and

$$\overline{p_i^2} = \frac{\lambda'^2 \rho_i^2}{2\pi}\int_A \varphi(0)\left[\frac{\rho\, d\rho}{r^2} - \frac{\rho^3\, d\rho}{r^4}\right]\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}$$

$$\overline{p_i^2} = \frac{\lambda'^2 \rho_i^2}{4\pi}\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\left[\frac{R^2}{R^2+Y^2}\right]$$

Here the plate has been assumed circular and R is its diameter.

APPENDIX II

THE SIMPLIFICATION OF THE PLATE RESPONSE INTEGRAL

(Equation 16)

We consider the approximation equation

$$\int_0^\infty \frac{\kappa\, d\kappa}{\left[\frac{c^2h^2}{\delta^2 U_\infty^2}\kappa^4 - \Omega^2\right]^2 + \left[\frac{\beta\delta}{U_\infty}\right]^2 \Omega^2} \cong \int_0^\infty \frac{\kappa\, d\kappa}{4\Omega^2\left[\frac{ch}{\delta U_\infty}\kappa^2 - \Omega\right]^2 + \left(\frac{\beta\delta}{U_\infty}\right)^2 \Omega^2} \tag{16}$$

The right-hand side is clearly unbounded as the damping constant $\beta \longrightarrow 0$ since its value is explicitly proportional to $1/\beta$ (see for instance Eq.(18)). On the other hand the difference between the left and the right-hand integrals is finite for $\beta = 0$. To show this we write

$$\xi = \frac{ch\kappa^2}{\delta U_\infty}$$

Then the left-hand side becomes for $\beta = 0$

$$\frac{\delta U_\infty}{ch\Omega} \int_0^\infty \frac{d\xi}{(\xi^2 - 1)^2} \tag{II1}$$

and the right-hand side becomes for $\beta = 0$

$$\frac{\delta U_\infty}{4ch\Omega} \int_0^\infty \frac{d\xi}{(\xi - 1)^2} \tag{II2}$$

Now

$$\frac{1}{(\xi^2 - 1)^2} = \frac{1}{4}\left\{\frac{1}{(\xi - 1)^2} + \frac{1}{(\xi + 1)^2} + \frac{1}{(\xi + 1)} - \frac{1}{(\xi - 1)}\right\} \tag{II3}$$

so that the difference D between expressions (II1) and (II2) is

$$D = \frac{\delta U_\infty}{2ch\Omega}\,\frac{1}{4}\int_0^\infty \left\{\frac{1}{(\xi + 1)^2} + \frac{1}{(\xi + 1)} - \frac{1}{(\xi - 1)}\right\}$$

or

$$D = \frac{\delta U_\infty}{8ch\Omega} \tag{II4}$$

This expression is finite and of course independent of β so that we can conclude that the left-hand integral of (16) is unbounded for $\beta = 0$. Further, it is clear that D is a regular function of β so that the ratio of the left-hand side to the right-hand side of Eq. (16) can be made arbitrarily close to unity, by choosing arbitrarily small β. If a correction is desired a numerical check indicates that Eq. (II4) gives a good approximation to the error made even with moderately large damping.

APPENDIX III

THE EVALUATION OF THE INTEGRAL SCALE

Our starting point is Eq. (11). In terms of non-dimensional variables it becomes

$$\lambda_x \overline{\left(\frac{\partial v_n}{\partial t}\right)^2} = \frac{\pi}{2}\,\frac{\rho_0^2 U_\infty^4 \delta}{\sigma^2 h^2}\left[\int_0^\infty \int_{-\infty}^\infty \frac{\Omega^4 F_2(\Omega, K_1, K_2)\, dK_2\, d\Omega}{\left[\frac{c^2h^2}{\delta^2 U_\infty^2}(K_1^2+K_2^2)^2-\Omega^2\right]^2 + \frac{\beta^2\delta^2}{U_\infty^2}\Omega^2}\right]_{K_1=0} \tag{III1}$$

We now simplify the denominator by writing successively

$$\left[\frac{c^2h^2}{\delta^2 U_\infty^2}(K_1^2+K_2^2)^2-\Omega^2\right]^2 = \left[\frac{ch}{\delta U_\infty}(K_1^2+K_2^2)+\Omega\right]^2\left[\frac{ch}{\delta U_\infty}(K_1^2+K_2^2)-\Omega\right]^2 \tag{III2}$$

Then we define

$$\frac{ch}{\delta U_\infty} = \sigma^2$$

$$(\Omega - \sigma^2 K_1^2) = \xi^2 \qquad \Omega > \sigma^2 K_1^2$$

$$(\Omega + \sigma^2 K_1^2) = \eta^2$$

Equation (III2) can now be written

$$(\sigma^2 K_2^2 + \eta^2)(\sigma K_2 + \xi)^2(\sigma K_2 - \xi)^2$$

and if we replace K_2 by its value at resonance,* namely

$$K_2 = \frac{\xi}{\sigma}$$

* The justification for that step is identical to that advanced in Appendix II.

We can write

$$\overline{\left(\frac{\partial V_n}{\partial t}\right)^2}\,\lambda_x = \frac{\pi}{2}\,\frac{\rho_0^2 U_\infty^4 \delta}{\sigma^2 h^2}\left[\int_0^\infty \Omega^4 F_2\left(K_1, \frac{\xi}{\sigma}, \Omega\right)\int_{-\infty}^{\infty}\frac{dK_2}{4(\xi^2+\eta^2)\xi^2(\sigma K_2-\xi)^2+\frac{\beta^2\delta^2}{U_\infty^2}\Omega^2}\right]_{K_1=0}$$

$$= \frac{\pi}{8}\,\frac{\rho_0^2 U_\infty^4 \delta}{\sigma^2 h^2}\left(\frac{ch}{\delta U_\infty}\right)^{-\frac{1}{2}}\frac{U_\infty}{\beta\delta}\int_0^\infty \Omega^{3/2} F_2\left(\sqrt{\frac{ch}{\delta U_\infty}}\,\Omega, \Omega\right) d\Omega \qquad \text{(III3)}$$

If we compare (III3) to (18) we get immediately

$$\lambda_x \cong \left(\frac{ch\delta}{U_\infty}\right)^{1/2}\frac{\int_0^\infty \Omega^{3/2} F_2\left(\sqrt{\frac{ch}{\delta U_\infty}}\,\Omega, \Omega\right) d\Omega}{\int_0^\infty \Omega^2 F\left(\sqrt{\frac{ch}{\delta U_\infty}}\,\Omega, \Omega\right) d\Omega}$$

$$= \frac{ch\delta}{U_\infty}\, f_2\left(\frac{\delta U_\infty}{ch}\right) \qquad \text{(III4)}$$